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The applicability of continuum models in the transitional regime of hypersonic flow over blunt bodies☆

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ABSTRACT

Hypersonic rarefied gas flow over blunt bodies in the transitional flow regime (from continuum to free-molecule) is investigated. Asymptotically correct boundary conditions on the body surface are derived for the full and thin viscous shock layer models. The effect of taking into account the slip velocity and the temperature jump in the boundary condition along the surface on the extension of the limits of applicability of continuum models to high free-stream Knudsen numbers is investigated. Analytic relations are obtained, by an asymptotic method, for the heat transfer coefficient, the skin friction coefficient and the pressure as functions of the free-stream parameters and the geometry of the body in the flow field at low Reynolds number; the values of these coefficients approach their values in free-molecule flow (for unit accommodation coefficient) as the Reynolds number approaches zero. Numerical solutions of the thin viscous shock layer and full viscous shock layer equations, both with the no-slip boundary conditions and with boundary conditions taking into account the effects slip on the surface are obtained by the implicit finite-difference marching method of high accuracy of approximation. The asymptotic and numerical solutions are compared with the results of calculations by the Direct Simulation Monte Carlo method for flow over bodies of different shape and for the free-stream conditions corresponding to altitudes of 75–150 km of the trajectory of the Space Shuttle, and also with the known solutions for the free-molecule flow regine. The areas of applicability of the thin and full viscous shock layer models for calculating the pressure, skin friction and heat transfer on blunt bodies, in the hypersonic gas flow are estimated for various free-stream Knudsen numbers.

Problems of hypersonic rarefied gas flow over blunt bodies are important when investigating the heat transfer and aerodynamics of re-entering spacecraft, and also apparatus performing manoeuvres in the upper layers of the Earth's atmosphere. In the transition from a continuum to a free-molecule flow regime, corresponding to high Knudsen numbers, of low Reynolds numbers, continuum flow models, such as the complete or parabolized Navier–Stokes equations, and also the equations of a full viscous shock layer, not to mention the boundary-layer equations, give values of the skin friction and heat transfer coefficients which increase without limit as the Reynolds number Re approaches zero, i.e., they give physically incorrect results. By taking into account the slip velocity and the temperature jump on the surface one can reduce the values of these coefficients and hence extend the range of applicability of these continuum models towards lower Re numbers, but one cannot eliminate the fact that these coefficients exceed the free-molecule limits for certain values of Re (taking into account the slip effects this value can be somewhat reduced) and their further infinite increase. Hence, in the transitional flow regime, to solve problems of hypersonic flow over bodies, the Direct Simulation Monte Carlo method is usually employed or kinetic models are used. In engineering practice, the approximate method of interpolating the heat-transfer and drug coefficients between their values in the free-molecule regime and the values in the boundary layer – the so-called bridge method, is also used. During the last ten-fifteen years different hybrid methods have been developed: these methods combine the solution of the kinetic equations – Boltzmann's equation or its simplified models – or the solution obtained by the Direct Simulation Monte Carlo method, with the solution of the continuum equations – the Navier–Stokes or asymptotically simplified models. Difficulties then arise connected with matching the different solutions. The various methods of investigating hypersonic flow of a rarefied gas over blunt bodies are reviewed in Ref.1. The limitation on the use of continuum models for investigating hypersonic rarefied gas flow at low Reynolds numbers does not exclude the use of a particular continuum approach for the correct prediction of certain flow parameters, such as the heat-transfer coefficient, the coefficient of skin friction and the pressure, in the transitional flow regime.

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In this paper, to determine these parameters, we have used two continuum models, namely, the full viscous shock-layer model and the thin viscous shock-layer model. These were proposed by Davis[4] and Cheng[2,3] respectively, and were later employed for regimes with Re>>1. They are discussed in Ref. 5. To investigate the justification for using these models at low Re numbers, an asymptotic analysis of the Navier–Stokes equations was carried out[1] for a shypersonic viscous shock layer over a blunt body at low Re numbers and it was shown that the two models are correct, i.e., they are asymptotically strictly derived from the Navier–Stokes equations not only for high Re numbers but also for low Re numbers, assuming the parameter introduced is small. As will be shown below, the asymptotically correct full viscous shock layer model, which takes all the second-order effects of boundary layer theory into account, proposes the use on the surface of boundary conditions which take into account the slip velocity and the temperature jumps, while the asymptotically correct thin viscous shock layer model proposes the use of the no-slip boundary conditions without temperature jump on the surface. In both models the Rankine–Hugoniot conditions on the required head shock are assumed.

As mentioned above, the applicability of the full viscous shock layer model is limited from below to low Re numbers, whereas the applicability of the thin viscous shock layer model is limited from above by virtue of the inaccuracy in determining the pressure on the body surface for some bodies, such as, for example, a sphere, at high Re numbers. At the same time, the thin viscous shock layer model gives correct values of the skin friction cofficient, the heat-transfer coefficient and the pressure at low Re numbers, and also correct free-molecule limits (for unit accommodation coefficient) for these coefficients as Re→0. The regions of applicability of the two models overlap: there is an intermediate region of Re numbers where each of the models gives results that are close and reliable.

In this paper, we use both the numerical method[6,7] and the asymptotic method[8,9] of investigation. By comparing the asymptotic and numerical solutions of full and thin various shock layer equations with the results of calculations obtained in the literature by the Direct Simulation Monte Carlo method, carried out for the continuum and transitional flow regimes, and also with the solutions in the free-molecule flow regime, it is shown that it is possible to use the continuum models to predict the heat flux and the skin friction on smooth blunt bodies for all hypersonic flow regimes, and that these methods are much more economic in computing costs than kinetic methods and Direct Simulation Monte Carlo methods.

1. The viscous shock layer (VSL) and thin viscous shock layer (TVSL) equations

Consider the steady supersonic and hypersonic laminar translational flow of a homogeneous viscous heat-conducting perfect gas over a plane or axisymmetric blunt body at zero angle of attack. The flow in the viscous shock layer between the head shock and the surface of the body, the contour of which is assumed to be fairly smooth (with a possible discontinuity in the curvature), will be considered in a curvilinear orthogonal system of coordinates (x,y), connected with its surface: the position of the point P in the flow is defined by its distance y to the contour of the body along the normal and the length x of the arc along the contour, measured from its vertex O to the base of the normal (Fig. 1).

The problem of hypersonic viscous flow of a rarefied gas over blunt bodies at low Reynolds numbers Re, or high Knudsen numbers Kn, is solved using two continuum models: the full viscous shock layer (VSL) model and the thin viscous shock layer (TVSL) model. The equations of the TVSL model were derived for the first time from the Navier–Stokes equations[2,3] for moderately high Re numbers with the following conditions

$$(\gamma-1)\,M_\infty^2 \gg 1, \quad \varepsilon \ll 1, \quad \mathrm{Re} \gg 1, \quad \varepsilon\mathrm{Re} = O(1)$$

where

$$\mathrm{Re} = \frac{\rho_\infty V_\infty R_0}{\mu_0}, \quad \varepsilon = \frac{\gamma-1}{2\gamma} \tag{1.1}$$

Here ε is the hypersonic parameter, γ is the ratio of the specific heats, M_∞, $\rho\infty$ and V_∞ are the free-stream Mach number, density and velocity, μ_0 is the viscosity coefficient at a temperature T_0, $T0 = V_\infty^2/(2c_p)$ (we can also choose another temperature as the characteristic temperature T_0, for example, the characteristic mean temperature in the shock layer), c_p is the specific heat at constant pressure and R_0 is the radius of curvature of the body surface at its vertex. The VSL model was proposed and justified in Ref. 4 also for high Re numbers.

In order to justify the correctness of using the VSL and the TVSL models at low Re numbers, an asymptotic analysis of the Navier-Stokes equations was carried out[1] in the problem of hypersonic flow over a blunt body at low Re numbers. This analysis enabled the presence of two important parameters of the problem to be detected. One of them is

$$\chi = 1/(\rho_s u_s) = O((\mu_s/(\rho_s \mathrm{Re}))^{1/2})$$

where the subscript s corresponds to values of the dimensionless parameters on the shock (ρ_s is the density relative to the free-stream density, u_s is the component of the velocity tangential to the shock, relative to the free-stream velocity and μ_s is the coefficient of viscosity relative to μ_0), and the shock layer thickness is of the order of χ. The second parameter is the rarefaction parameter

$$\tau = u_s = O((\mathrm{Re}/(\mu_s \rho_s))^{1/2})$$

The dimensionless tangential velocity and enthalpy on the shock are of the order of τ.

An asymptotic analysis of the Navier-Stokes equations showed that the TVSL and VSL equations hold not only for high Re numbers but also for low Re numbers (the small parameter τ), assuming that the parameter χ is small: $\chi << 1$. The VSL equations are obtained from the Navier-Stokes equations if we neglect terms $O(\chi^2)$ and take terms $O(\chi)$ into account. The TVSL equations are obtained from the Navier-Stokes equations if we neglect terms $O(\chi^2)$ and $O(\chi)$, with the exception of the term with the longitudinal pressure gradient, which is of the order of χ, i.e., out-of-order of magnitude, and remains in the TVSL equations, since it plays an important role at high Re numbers. The asymptotically correct TVSL model consists of equations without the out-of-order of magnitude term with a longitudinal pressure gradient in the first (marching) impulse equation; just this model gives correct limiting values for the skin friction coefficient when Re→0.

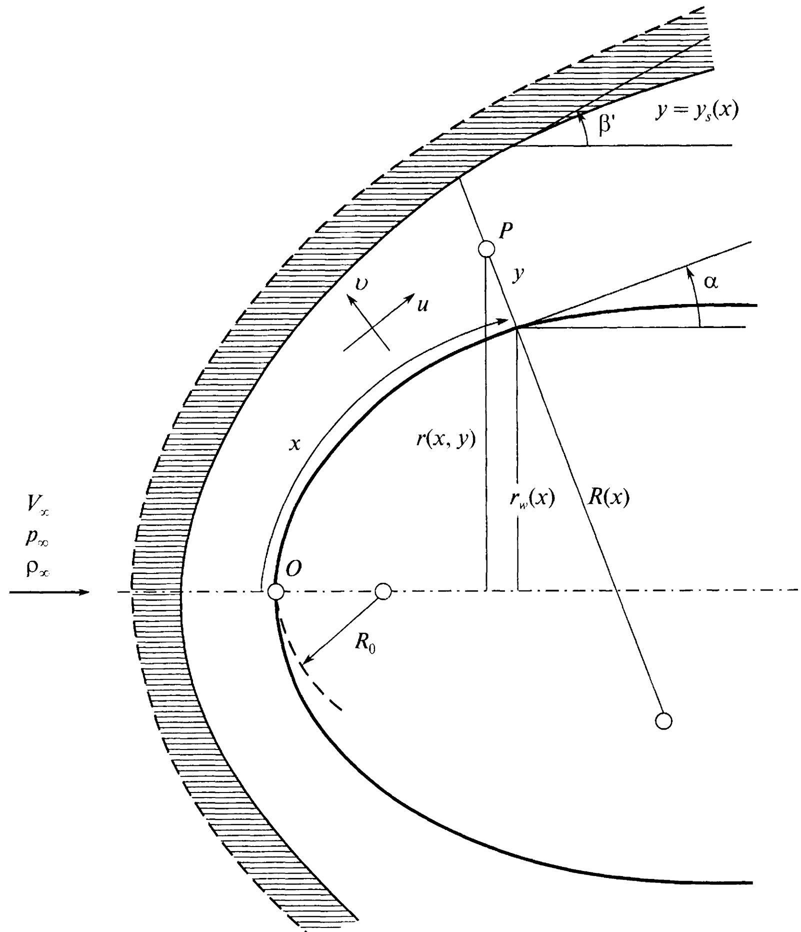

Fig. 1.

The VSL equations in the orthogonal system of coordinates (x,y), which is connected in a natural way with the surface of the body in the flow field (Fig. 1), take the form

$$\frac{\partial}{\partial x}\left(r^\nu \rho u\right) + \frac{\partial}{\partial y}\left(H_1 r^\nu \rho \upsilon\right) = 0$$
$$\rho\left(\frac{u}{H_1}\frac{\partial u}{\partial x} + \upsilon\frac{\partial u}{\partial y} + \frac{u\upsilon}{RH_1}\right) + \frac{1}{H_1}\frac{\partial p}{\partial x} = \frac{1}{H_1^2 r^\nu \mathrm{Re}}\frac{\partial}{\partial y}\left[H_1^2 r^\nu \mu\left(\frac{\partial u}{\partial y} - \frac{u}{RH_1}\right)\right]$$
$$\rho\left(\frac{u}{H_1}\frac{\partial \upsilon}{\partial x} + \upsilon\frac{\partial \upsilon}{\partial y} - \frac{u^2}{RH_1}\right) + \frac{\partial p}{\partial y} = 0$$
$$\rho\left(\frac{u}{H_1}\frac{\partial H}{\partial x} + \upsilon\frac{\partial H}{\partial y}\right) = \frac{1}{H_1 r^\nu \mathrm{Re}\,\mathrm{Pr}}\frac{\partial}{\partial y}\left\{H_1 r^\nu \mu\left[\frac{\partial H}{\partial y} - (1-\mathrm{Pr})\frac{\partial u^2}{\partial y} - 2\mathrm{Pr}\frac{u^2}{RH_1}\right]\right\}$$
$$H = T + \left(u^2 + \upsilon^2\right) \quad (1.2)$$

Here $r(x) = r_w + y\cos\alpha$ is the distance from a point in the flow to the axis of symmetry, $r_w(x)$ is the distance from a point on the body contour to the axis of symmetry, $\alpha(x)$ is the angle between the tangent to the body contour and the axis of symmetry, $H_1 = 1 + y/R$ is the Lamé coefficient, R(x) is the radius of curvature of the body contour, $V_\infty u$ and $V_\infty v$ are the components of the velocity vector along the *x* and *y* axes respectively, $\rho_\infty \rho$ is the density, $\mu_0\,\mu$ is the coefficient of viscosity, $\rho_\infty V_\infty^2 p$ is the pressure, $T_0 T$ is the temperature $HV_\infty^2/2$ is the total enthalpy and Pr is the Prandtl number; $\nu = 0$ for the plane problem and $\nu = 1$ for the axisymmetric problem. All the quantities with the dimension of length are relative to R_0.

System of equations (1.1) is supplemented by the equation of state of a perfect gas, and also by the relation which defines the temperature dependence of the coefficient of viscosity, which, in this paper is assumed to be a power function with exponent ω:

$$\rho = \frac{p}{\varepsilon T}, \quad \mu = T^\omega \quad (1.3)$$

The TVSL equations have the form

$$\frac{\partial}{\partial x}\left(r_w^\nu \rho u\right)+\frac{\partial}{\partial y}\left(r_w^\nu \rho \upsilon\right)=0$$
$$\rho\left(u\frac{\partial u}{\partial x}+\upsilon\frac{\partial u}{\partial y}\right)+\frac{\partial p}{\partial x}=\frac{1}{\mathrm{Re}}\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)$$
$$\frac{\partial p}{\partial y}-\frac{\rho u^2}{R}=0$$
$$\rho\left(u\frac{\partial H}{\partial x}+\upsilon\frac{\partial H}{\partial y}\right)=\frac{1}{\mathrm{Re}\,\mathrm{Pr}}\frac{\partial}{\partial y}\left\{\mu\frac{\partial}{\partial y}\left[H-(1-\mathrm{Pr})u^2\right]\right\}$$
$$H=T+u^2 \tag{1.4}$$

2. The boundary conditions on the shock

We have also carried out an asymptotic analysis[1] of the generalized Rankine–Hugoniot conditions on the shock for low Re numbers. The boundary conditions on the outer boundary of the shock layer for the TVSL are obtained, if we neglect terms $O(\chi^2)$ and $O(\chi)$, and the boundary conditions for the VSL are obtained, if we only neglect terms $O(\chi^2)$.

The boundary conditions on the shock $y = y_s(x)$ for the VSL equations have the form

$$\upsilon_s=u_s\,\mathrm{tg}\beta_s-\frac{1}{\rho_s}\frac{\sin\beta'}{\cos\beta_s},\quad \beta_s=\beta'-\alpha$$
$$u_s=\cos\beta'\cos\beta_s+\frac{1}{\rho_s}\sin\beta'\sin\beta_s-\frac{\mu_s\cos^3\beta_s}{\mathrm{Re}\sin\beta'}\left(\frac{\partial u}{\partial y}-\frac{u}{RH_1}\right)_s$$
$$H_s=1+\frac{2}{(\gamma-1)\mathrm{M}_\infty^2}-\frac{\mu_s\cos\beta_s}{\mathrm{Re}\,\mathrm{Pr}\sin\beta'}\left[\frac{\partial H}{\partial y}-(1-\mathrm{Pr})\frac{\partial u^2}{\partial y}-2\mathrm{Pr}\frac{u^2}{RH_1}\right]_s$$
$$p_s=\left(1-\frac{1}{\rho_s}\right)\sin^2\beta'+\frac{1}{\gamma\mathrm{M}_\infty^2} \tag{2.1}$$

Here $\beta'(x)$ is the angle of inclination of the shock to the axis of symmetry (Fig. 1), $\beta_s(x)$ is the angle of inclination of the shock to the x axis; the angle β_s and the shock detachment y_s are related by the geometrical relation

$$dy_s/dx=H_{1s}\,\mathrm{tg}\beta_s \tag{2.2}$$

It should be noted that in the VSL model proposed by Davis[4], as in many subsequent papers, which use this model to solve different problems, in the boundary conditions for the tangential velocity and the enthalpy the factors $\cos^3\beta_s$ and $\cos\beta_s$ in front of the square brackets are not present, which is obviously due to the fact that the difference in the directions of the normal to the body and the normal to the shock wave is ignored. In our investigation the calculations were carried out using the classical model of the VSL,[4] with approximate conditions on the shock, but it is more correctly to solve the VSL equations with conditions (2.1). Note also that although the VSL equations (1.2) themselves, obtained for high and low Re numbers, are completely identical, there are some differences in the boundary conditions on the shock wave for low and high Re numbers: thus, for low Re numbers in the boundary conditions for the tangential velocity the term with $1/\rho_s$ is out of order of magnitude. Hence, the question of the effect of different terms in the boundary conditions on the solution and of the formulation of the correct boundary conditions for various Re numbers, should be investigated.

The exact relation (2.2), which relates dy_s/dx and $\beta_s = \beta' - \alpha$ in the VSL model, is replaced in the TVSL model by the approximate relation $\beta' \approx \alpha$. The asymptotically correct boundary conditions on the shock for the TVSL model will be

$$\upsilon_s=u_s\frac{dy_s}{dx}-\frac{1}{\rho_s}\sin\alpha,\quad u_s=\cos\alpha-\frac{\mu_s}{\mathrm{Re}\sin\alpha}\left(\frac{\partial u}{\partial y}\right)_s$$
$$H_s=1-\frac{\mu_s}{\mathrm{Re}\,\mathrm{Pr}\sin\alpha}\frac{\partial}{\partial y}\left[H-(1-\mathrm{Pr})u^2\right]_s,\quad p_s=\sin^2\alpha \tag{2.3}$$

Note that, in the first of relations (2.3), it is necessary to take into account the small term with dy_s/dx, because all the terms of this equation are infinitesimals of the same order. This is a point of principle. In an asymptotic analysis small terms are neglected compared with terms of the order of unity, but if all the terms in the equation are of the same order, although small, they must all be taken into account. And here we should note the incorrectness of Cheng's approach[2,3] (also repeated in later papers on the TVSL), who wrote this boundary condition – the first relation in (2.1) – ignoring the term with the derivative dy_s/dx. This relation is in fact the equation of conservation of mass on the shock, and if this term is ignored, a contradiction is obtained between this equation and the continuity equation in system (1.4). Here the correct representation of the boundary conditions gives rise to no additional difficulties in the calculations.

3. The boundary conditions on the surface of the body

We will analyse the boundary conditions on the surface of the body for low Re numbers. We will write the boundary conditions on the surface, taking into account the slip velocity and the temperature jump in the variables assumed above (different variations of these boundary conditions exist, and they have been considered in numerical calculations, but the asymptotic estimates for these are exactly the same):[10]

$$u = \frac{2-\theta}{\theta}\sqrt{\frac{\pi}{2\varepsilon^{1/2}}}\frac{\mu}{\mathrm{Re}T^{1/2}\rho}\frac{\partial u}{\partial y}$$
$$T = T_w + \frac{2-\alpha'}{\alpha'}\frac{2\gamma}{\Pr(\gamma+1)}\sqrt{\frac{\pi}{2\varepsilon^{1/2}}}\frac{\mu}{\mathrm{Re}T^{1/2}\rho}\frac{\partial T}{\partial y} \tag{3.1}$$

Here θ and α' are the diffuse reflection coefficient and the accommodation coefficient (they were taken to be equal to unity in the calculations), and T_wT_0 is the temperature of the body surface.

To carry out an asymptotic analysis we will convert boundary conditions (3.1) to a new independent variable η, in other words, we will write them in Dorodnitsyn variables (just in these variables the asymptotic analysis of the equations and boundary conditions on a shock was carried out in Ref. 1)

$$\eta = \frac{1}{\Delta}\int_0^y \rho dy, \quad \Delta = \Delta(x) = \int_0^{y_s(x)} \rho dy \tag{3.2}$$

The corresponding conversion of conditions (3.1) to the new independent variable η leads to the fact that $\frac{1}{\rho}\frac{\partial}{\partial y}$ is replaced by $\frac{1}{\Delta}\frac{\partial}{\partial \eta}$.

We will use the following estimates,[1] made for low Re numbers, for the asymptotic analysis of the coefficients in the converted conditions (3.1):

$$u = O(\tau), \quad \frac{T-T_w}{1-T_w} = O(\tau), \quad \frac{1}{\rho} = O(\chi\tau), \quad \Delta^{-1} = O(\tau), \quad \frac{\mu}{\rho \mathrm{Re}} = O(\chi^2) \tag{3.3}$$

and we obtain estimates of the slip velocity and the in temperature jump on the surface

$$u = \frac{O(\chi)}{\varepsilon^{1/2}T^{1/2}}\frac{\partial u}{\partial \eta}, \quad T - T_w = \frac{O(\chi)}{\varepsilon^{1/2}T^{1/2}}\frac{\partial T}{\partial \eta} \tag{3.4}$$

or

$$u = \frac{O(\chi)}{\varepsilon^{1/2}}O(\tau^{1/2+k}), \quad T - T_w = \frac{O(\chi)}{\varepsilon^{1/2}}O(\tau^{1/2+k}), \quad 0 \le k \le \frac{1}{2} \tag{3.5}$$

Here k = 0 for a cold wall (regime I) and k = ½ for regime III (we will discuss the flow regimes in the next section).

The VSL equations are derived neglecting terms $O(\chi^2)$ and taking terms $O(\chi)$ into account, and hence it is obvious that in this model we must take into account the slip effects and a temperature jump on the surface, since these effects are of the some order of χ as the terms of the equations. The TVSL equations are derived neglecting terms both $O(\chi^2)$ and $O(\chi)$, and hence the asymptotically correct model of the TVSL is obtained neglecting the slip effects on the surface, since these effects are of the order of χ. Note that the TVSL model with the no slip conditions on the surface gives correct free-molecule limits (with an accommodation coefficient equal to unity) for the heat-transfer coefficient and the skin friction coefficient as Re→0. Numerical calculations show that when using the slip boundary conditions the TVSL model as Re→0 gives reduced limit values of these coefficients.

Hence, the VSL model used in this paper includes Eqs (1.2), the generalized Rankine–Hugoniot relations on the shock (2.1) together with condition (2.2) and the slip boundary conditions (3.1) together with the non-flow condition v = 0 on the surface of the body. The model of the TVSL employed includes Eqs. (1.4), the generalized Rankine–Hugoniot relations on the shock (2.3) and the no-slip conditions, and the assumption that the gas temperature is equal to the wall temperature on the surface:

$$u = 0, \quad \upsilon = 0, \quad T = T_w \tag{3.6}$$

The heat-transfer coefficient and the skin friction coefficient on the surface are defined as follows:

$$c_f = \frac{2\tau_w}{\rho_\infty V_\infty^2}, \quad \tau_w = \left(\mu_0\mu\frac{\partial(V_\infty u)}{\partial y}\right)_w$$
$$c_H = \frac{q}{\rho_\infty V_\infty (H_\infty - H_w)}, \quad q_w = \left(\lambda\frac{\partial(T_0 T)}{\partial y} + \mu\mu_0 V_\infty u\frac{\partial(V_\infty u)}{\partial y}\right)_w \tag{3.7}$$

4. The flow parameters

Consider the basic dimensionless parameters of the hypersonic flow of a rarefied gas over blunt bodies at high Kn_∞ numbers, or low Re numbers – χ and τ.

The parameters χ and τ are found from the values of the flow parameters on the shock, and they can be expressed in terms of the governing parameters of the problem Re, ε, T_w, Pr, ω ($\mu = T^\omega$) and the geometrical parameters of the body in the flow field occurs. In order to estimate χ and τ we must estimate u_s, ρ_s and μ_s. This can be done by an asymptotic analysis of relations (2.3) on the shock, where the estimates of these quantities depend on the regime of hypersonic rarefied gas flow over the body. The regimes of hypersonic rarefied gas flow over the bodies were considered for the neighbourhood of the stagnation line[1] and for the windward part of axisymmetric and plane bodies.[8] In the case of two-dimensional flows we can distinguish three flow regimes when $\varepsilon\text{Re} = o(1)$:

$$\text{I}\ \ \varepsilon\text{Re} \gg \frac{T_w^{1+\omega}}{\beta},\quad \text{II}\ \ \varepsilon\text{Re} = O\left(\frac{T_w^{1+\omega}}{\beta}\right),\quad \text{III}\ \ \varepsilon\text{Re} \ll \frac{T_w^{1+\omega}}{\beta};\quad \beta = \frac{1}{2}\left(\frac{\sin\alpha}{R} + \nu\frac{\sin\alpha\cos\alpha}{r_w}\right)$$

The geometrical parameter $\beta \sim 1$ at the stagnation point. An asymptotic analysis shows that when $\varepsilon\text{Re} = o(1)$

$$u_s,\ g_s,\ T_s = O((\varepsilon\text{Re}\beta)^{1/(1+\omega)}) \text{ in regims Iand II}$$

$$u_s,\ g_s = O((\varepsilon\text{Re}T_w^{1-\omega}\beta)^{1/2}) \text{ in regims III},\ T_s \sim T_w$$

In regime I (the case of a strongly cooled surface) the temperature factor T_W droops out of the governing parameters of the problem. We obtain the following estimates for the parameters τ and χ

$$\chi = O(\varepsilon(\beta/\sin^2\alpha)^{1/2}),\quad \tau = O((\varepsilon\text{Re}\beta)^{1/(1+\omega)}) \text{ in regims Iand II}$$

$$\chi = O((\varepsilon T_w^{1+\omega}/(\text{Re}\sin^2\alpha))^{1/2}),\quad \tau = O((\varepsilon\text{Re}T_w^{1-\omega}\beta)^{1/2}) \text{ in regims III}$$

The parameter τ represents the rarefaction of the flow, and also the applicability of the asymptotic solution, which we will discuss below. When $\beta = 1$ the parameter τ depends on the flow parameters ε, Re, ω and T_W and represents the rarefaction of the flow in the neighbourhood of the stagnation point (at this point $\beta = (1+\nu)/2$); due to the geometrical parameter β the parameter τ increases as the distance from the forward stagnation point increases. The parameter χ represents the applicability of the continuum models in the transitional flow regime. In regimes I and II the parameter $\chi = O(\varepsilon)$, and in hypersonic flow it is small in the stagnation region. Hence, for a sufficiently cold wall, even for low Re numbers, the continuum models can be used to calculate a number of flow parameters. In regime III the parameter χ depends on Re, ε, ω, T_W.

5. The asymptotic solution

The two-dimensional TVSL equations (1.4) with the modified Rankine–Hugoniot relations (2.3) on the shock and with the no-slip boundary conditions on the body (3.6), were solved (for the correct TVSL model) using the integral method of successive approximations and the asymptotic expansion of the solution in series assuming the parameter εRe or τ to be small.[9] The analytic solutions were obtained for the heat-transfer coefficient c_H, the skin friction coefficient c_f and the pressure coefficient on the wall p_W for the three flow regimes in

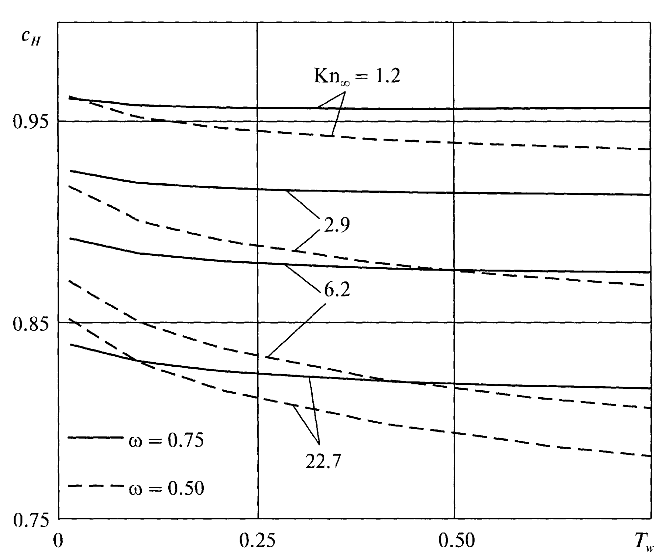

Fig. 2.

the case of asymmetric problems ($\nu = 1$) and plane problems ($\nu = 0$). For regime I (a cold wall) the solution has the form

$$
\begin{aligned}
c_H &= \sin\alpha\left[1 - \frac{1+\omega}{3(2-\omega)}\Pr\tau\right] + O(\tau^2),\\
c_f &= 2\sin\alpha\cos\alpha\left[1 - \frac{1}{3}\left(\frac{1+\omega}{2-\omega} + \frac{\sin\alpha}{R\beta}\right)\tau\right] + O(\tau^2)\\
\tau &= (\Pr^{1-\omega}\varepsilon\mathrm{Re}\beta)^{1/(1+\omega)},\quad \beta = \frac{1}{2}\left(\frac{\sin\alpha}{R} + \nu\frac{\sin\alpha\cos\alpha}{r_w}\right)
\end{aligned}
\tag{5.1}
$$

This solution represents the simple analytic dependence of the coefficients c_H and c_f on the free-stream parameters Re, ε, Pr, ω and the geometrical parameters of the body in the flow field α, r_w and R.

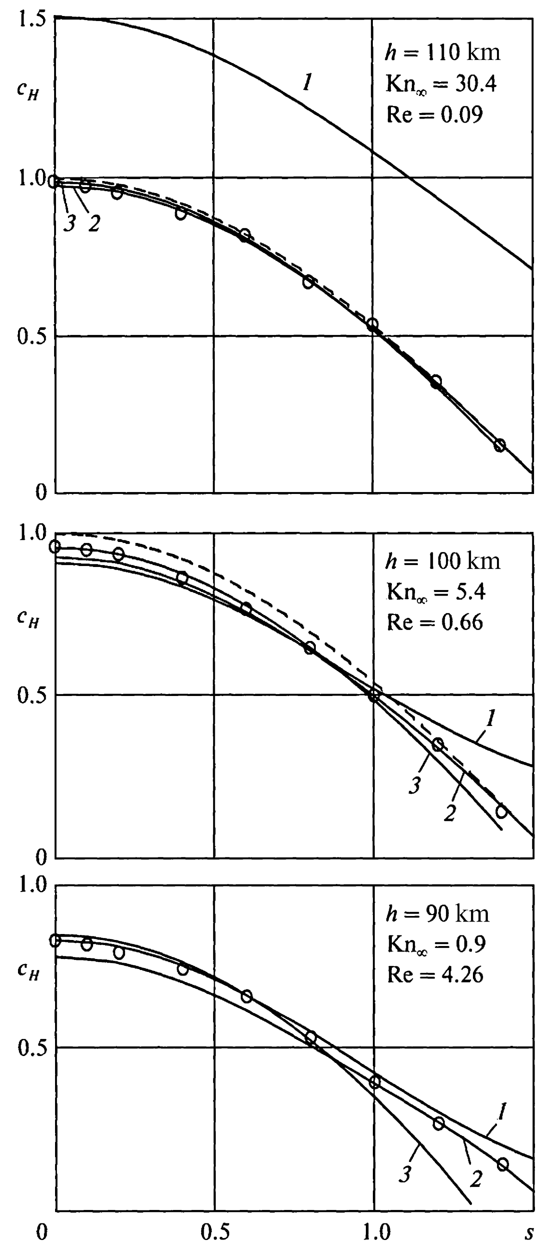

Fig. 3.

In the case of regimes II and III the solution also depends on the dimensionless surface temperature T_w. For regime II we have

$$c_H = \sin\alpha\left[1-\left(\phi-\frac{1}{3}\right)\Pr\tau\right]+O(\tau^2),$$
$$c_f = 2\sin\alpha\cos\alpha\left[1-\left(\phi+\frac{\sin\alpha}{3R\beta}-\frac{1}{3}\right)\tau\right]+O(\tau^2)$$
$$\phi=\frac{(1+\lambda)^{2-\omega}-\lambda^{2-\omega}}{(2-\omega)(1+\lambda)^{1-\omega}},\quad T_w=(\Pr^2\varepsilon\mathrm{Re}\beta)^{1/(1+\omega)}\lambda(1+\lambda)^{(1-\omega)/(1+\omega)}$$
$$\tau=(\Pr^{1-\omega}\varepsilon\mathrm{Re}\beta)^{1/(1+\omega)}(1+\lambda)^{(1-\omega)/(1+\omega)} \tag{5.2}$$

For regime III the solution is given by the first two equalities of (5.2) with $\phi\equiv 1$ and $\tau=(\varepsilon Re T_w^{1-\omega}\beta)^{1/2}$.

The pressure coefficient on the wall p_w for all three flow regimes is given by the formula (each regime has its own parameter τ)

$$p_w=\sin^2\alpha-\frac{\sin\alpha\cos^2\alpha}{3R\beta}\tau+O(\tau^2) \tag{5.3}$$

As the Re number approaches zero (as $\tau\to 0$, or $\varepsilon\mathrm{Re}\to 0$ the values of the heat-transfer coefficient, the skin friction coefficient and the pressure coefficient approach their values in free-molecule flow for unit accommodation coefficient[11]

$$\lim_{\mathrm{Re}\varepsilon\to 0} c_H=\sin\alpha,\quad \lim_{\mathrm{Re}\varepsilon\to 0} c_f=2\sin\alpha\cos\alpha,\quad \lim_{\mathrm{Re}\varepsilon\to 0} p_w=\sin^2\alpha \tag{5.4}$$

When $\omega = 1$ the solutions are identical for all three regimes, and

$$\tau=(\varepsilon\mathrm{Re}\beta)^{1/2} \tag{5.5}$$

For a cold surface, corresponding to regime I $T_w \ll (\varepsilon Re/\beta)^{1/(1+\omega)}$) the solution for the heat-transfer coefficient, the skin friction coefficient and the pressure coefficient is independent of the wall temperature T_w. (Regime I is the limiting case of regime II as $\lambda\to 0$.) The region of definition of regimes II and III are $Re\sim(\beta T_w)^{1+\omega}/\varepsilon$ and $Re \ll (\beta T_w)^{1+\omega})/\varepsilon$), and here the solutions for different regimes are very close to one another, approaching a common limit as $\mathrm{Re}\to 0$. When $\omega = 1$ the solutions for all three regimes are strictly identical and are independent of T_w, i.e., one can use the solution for a cold wall (5.1).

In Fig. 2 we show the results of numerical calculations of the TVSL for the heat-transfer coefficient at the forward stagnation point of an axisymmetric body (with a nose radius $R_0 = 1.36$ m) as a function of on the surface temperature T_w for different values of the number free-stream Knudsen $Kn_\infty = 1.2$, 2.9, 6.2 and 22.7, corresponding to different flight altitudes of the Space Shuttle h = 115, 122.5, 130 and 150 km. When $\omega = 0.75$ the dependence of the solution on T_w is quite weak, and the difference of the solutions for $T_w < 0.2$ from the solution for $T_w = 0$ varies from 0.5% to 1.5% for different altitudes. When $\omega = 0.5$ this difference varies from 1.5% to 4%, and when $T_w < 0.1$, which corresponds to the actual problems of the spacecraft motion in the upper Earth's atmosphere, up to 2.5%. Hence, for the problems considered, at high altitudes, where the asymptotic solution holds, one can use the cold-wall solution (5.1) with fairly good accuracy.

When $\omega = 1$ the similarity parameter τ is identical with Cheng's parameter $K = (\varepsilon\mathrm{Re})^{1/2}$,[2,3] with the exception that the parameter τ also includes the geometrical parameter β, i.e., it takes into account the effect of the geometry of the body. However, at the stagnation point of

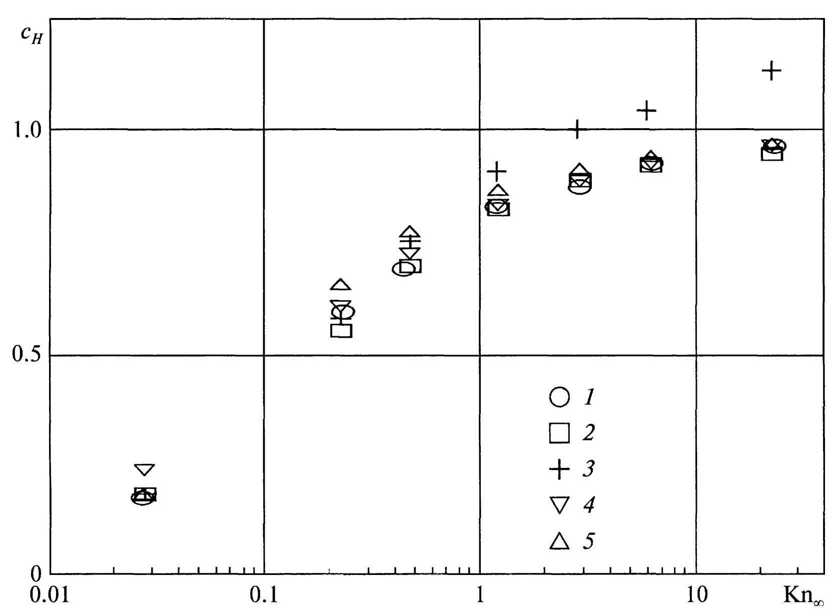

Fig. 4.

an axisymmetric body the parameter τ and Cheng's parameter coincide completely when ω = 1. In the general case, the parameter τ, unlike Cheng's parameter, in addition to the geometry of the body, also takes into account the effect of the ω value and the wall temperature T_w.

6. Numerical solution of the VSL and TVSL equations

Discussion of the results. To solve the problem considered, in addition to the asymptotic method of investigation, we also used a numerical method. The two-dimensional VSL and TVSL equations, both with the no-slip boundary conditions on the surface and with boundary conditions which take into account slip and temperature jumps, were solved using an effective implicit finite-difference marching method with a high accuracy of the difference approximation (to the fourth power in the normal coordinate). This method is based on global iterations of only one function – the elliptic part of the difference approximation of the longitudinal (marching) pressure gradient – and requires an extremely small number (two-three) of global iterations with respect to the elliptic component of the longitudinal pressure gradient. We used a special splitting of the longitudinal pressure gradient into elliptic and hyperbolic components.[6,7]

To estimate the region of applicability of the models used to investigate the transitional flow regime (from continuum to free-molecule flow) we carried out systematic comparisons of the asymptotic and numerical solutions of the VSL and TVSL for the heat-transfer coefficient and the skin friction coefficient with results of calculations available in the literature using the Direct Simulation Monte Carlo (DSMC) method for bodies of different shape, and also with the solution in free-molecule flow. The free-stream parameters corresponded to the trajectory of the Space Shuttle reentry into Earth's atmosphere at altitudes of h = 75–150 km at V_∞ = 7.5 km/s, i.e., it covered the continuum and transitional flow regimes. Some results of the comparison are shown in Figs 3–6.

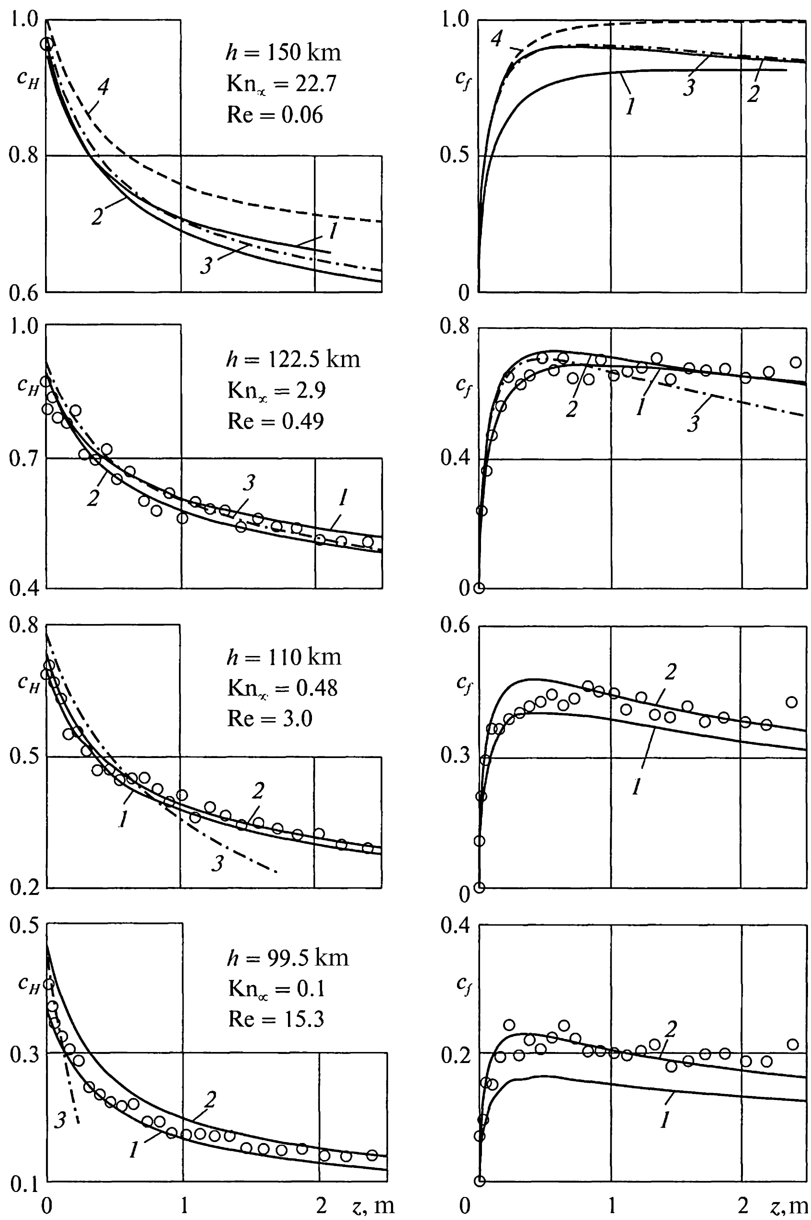

Fig. 5.

In Fig. 3 we show the distributions of the heat-transfer coefficient over the surface of a sphere of radius $R_0 = 0.0254$ m at altitudes of h = 90, 100 and 110 km, obtained from numerical solutions of the VSL and the TVSL (curves 1 and 2), the analytical solution (curves 3), the free-molecule solution (the dashed curve) and calculations using the DSMC method (the points).[12] It can be seen that taking the slip effects of on the wall into account extends the region of applicability of the VSL up to h = 100 km and $Kn_\infty = 5.4$. However, when the altitude is increased to h ∼ 110 km, and $Kn_\infty \sim 30$, the VSL model ceases to work, whereas the TVSL gives results identical with the solution by the DSMC method[12] and close to the free-molecule solution.

A comparison of the different methods of calculating the heat transfer at the stagnation point of the body with a nose radius $R_0 = 1.36$ m, along the trajectory of the Space Shuttle at altitudes of h = 92, 99.5, 105, 110, 115, 122.5, 130 and 150 km, corresponding to $Kn_\infty = 0.03$, 0.1, 0.23, 0.48, 1.2, 2.9, 6.2 and 22.7, is presented in Fig. 4: 1 is for calculations using the DSMC method,[13], 2 is for VSL with slip, 3 is for VSL without slip, 4 is for TVSL, and 5 is the asymptotic solution. This figure also demonstrates the influence of slip effects on the surface. With no-slip conditions and for a gas temperature equal to the wall temperature, the VSL equations give correct results for the heat-transfer coefficient at the stagnation point at altitudes less than 105 km or $Kn_\infty < 0.23$. Accounting for the slip velocity and a temperature jump extends the range of applicability of the VSL model for calculating the heat transfer at the stagnation point with $R_0 \sim 1$ m up to altitudes of 150 km and $Kn_\infty = 22.7$. The TVSL model – the numerical and asymptotic solutions – gives correct results at altitudes above 100 km, and the accuracy of these solutions increases with altitude.

In Fig. 5 we compare the distributions of the heat-transfer coefficient and the skin friction coefficient along the surface of a 42.5° hyperboloid with a nose radius $R_0 = 1.362$ m at altitudes of h = 150, 122.5, 110 and 99.5 km, obtained from the solution of the continuum equations – VSL (curves 1), TVSL (curves 2) and the asymptotic solution (curves 3) the calculations were carried out for $\gamma = 1.23$, $\omega = 0.75$ and Pr = 0.7), with the results obtained by the DSMC method[13] (the points) and the solution in the free-molecule flow regime (the dashed curve 4). A similar comparison of the continuum solutions ($\gamma = 1.25$, $\omega = 0.73$ and Pr = 0.7) with the solutions obtained by the DSMC method,[14] for distributions of the heat-transfer coefficient and the skin friction coefficient along the surface of a 50° hyperboloid with $R_0 = 1.143$ m at altitudes of h = 140, 110 and 95 km are shown in Fig. 6.

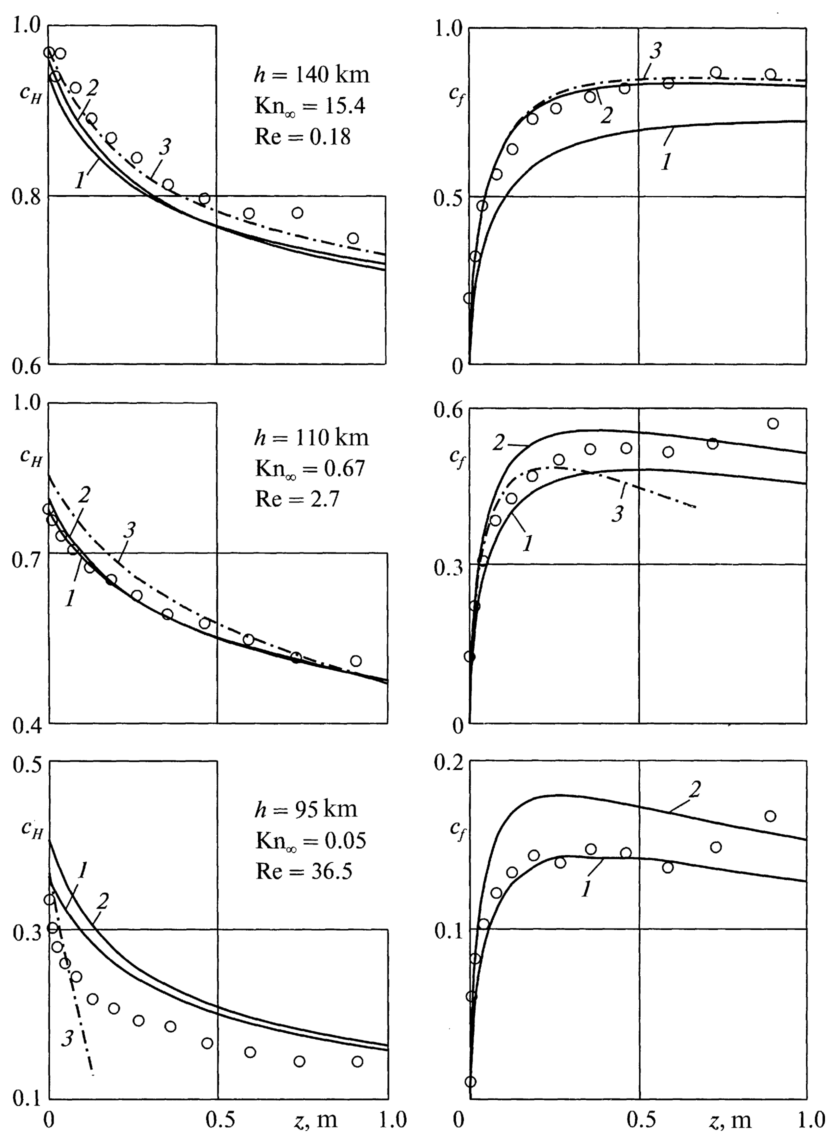

Fig. 6.

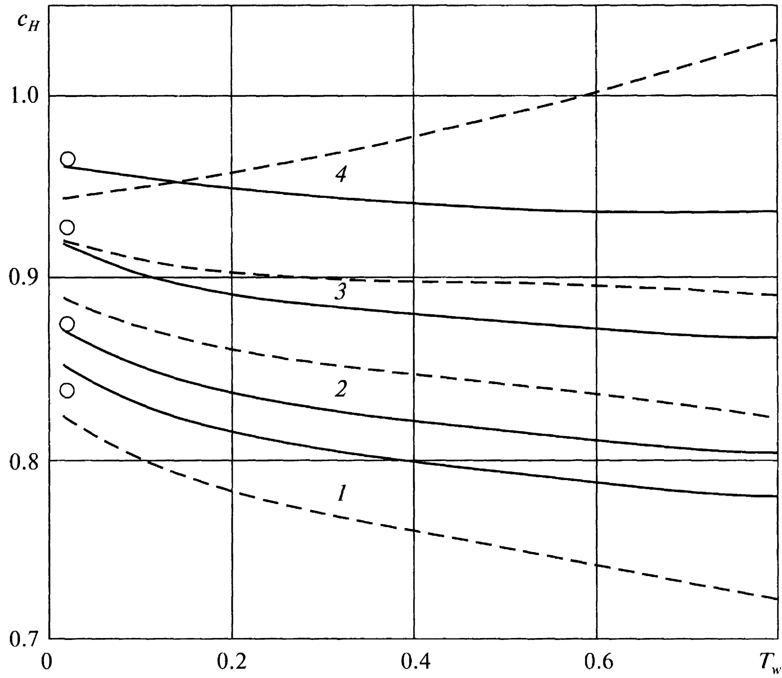

Fig. 7.

The above comparisons demonstrate that the VSL model, taking into account the slip effects, gives reliable results for a distribution of the heat flux up to an altitude of 150 km (for $R_0 = 1.362$ m) and $Kn_\infty = 22.7$ (Re = 0.06) not only at the stagnation point, but also along the side surface up to z = 2 m (the distance along the axis of the hyperboloid from its vettex). As regards the skin friction, the VSL gives somewhat reduced results compared with the DSMC method. Possibly this is due to the use of not entirely correct boundary conditions at the shock for the VSL model, particularly for the tangential component of the velocity, as was mentioned in Section 2. The numerical TVSL solution in all the cases considered makes good predictions for the heat-transfer coefficient and the skin friction coefficient at altitudes above 100 km (for $R_0 \approx 1$ m), or $Kn_\infty \geq 0.1$.

Figs. 3–6 demonstrate that the asymptotic solutions holds at $Kn_\infty \geq 0.1$ for c_H and at $Kn_\infty \geq 0.5$ for c_f in the region of the stagnation point. Far from the stagnation point it holds at $Kn_\infty \geq 1$ for c_H and at $Kn_\infty \geq 5$ for c_f. On the whole, the validity of the asymptotic solution depends not only on the values of Kn_∞, or the Re number, or the parameter τ, but also on the distance from the stagnation point, which is due to the fact that, for a long body (such as a hyperboloid), as the distance from the stagnation point increases the assumption that τ is small begins to be violated due to the effect of the geometrical parameter β, and the asymptotic solution becomes invalid. The greater the flight altitude (the greater the rarefaction and the less Re), the further the distances from the stagnation point at which the asymptotic solution holds. As the flight altitude increases, as $Kn_\infty \to \infty$, this solution approaches the solution in free-molecule flow.

The calculations and comparisons presented above correspond to a cold surface $0.02 \leq T_w \leq 0.07$, since this surface temperature corresponds to the actual spacecraft flight conditions and the calculations by the DSMC method available in the literature. In order to estimate the effect of the surface temperature on the heat flux and the applicability of the continuum models in the transitional flow regime, we calculated the VSL equations with slip (the dashed curves) and the TVSL equations (the continuous curves) at the forward stagnation point ($R_0 = 1.362$ m, $\gamma = 1.23$ and $\omega = 0.5$) at different trajectory altitudes h = 115, 122, 130 and 150 km corresponding to $Kn_\infty = 1.2$, 2.9, 6.2 and 22.7 (curves 1 - 4) for various surface temperatures, the results of which are shown in Fig. 7. The figure also shows the results of calculations by the DSMC method[13] (the points). It can be seen that whereas at $Kn_\infty < 10$ both models give close, qualitatively similar results, irrespective of the value of T_w, when $Kn\infty \sim 20$ the VSL model is only usable for a cold wall ($T_w < 0.1$), giving physically incorrect results for large T_w, i.e., the limits of the region of applicability of the VSL equations depends on the surface temperature.

The above comparisons enables us to draw the conclusion that, for a sufficiently cold surface, the VSL model gives reliable results for the heat flux and skin friction at $Kn_\infty \leq 10$, and the accuracy of this model increases as Kn_∞ decreases, while the TVSL model gives reliable results at $Kn_\infty \geq 0.1$, and in the limit as $Kn_\infty \to \infty$ the TVSL solution is identical with the solution in free-molecule flow. In the range $0.1 \leq Kn_\infty \leq 10$ both models give reliable results; hence, we can use the VSL model as the more accurate model for low Kn_∞ numbers, and when $Kn_\infty \sim 1$ we can change to the TVSL model, which is more correct for high Kn_∞ numbers for predicting the heat flux and skin friction on blunted bodies, in the hypersonic gas flow.

The results obtained provide grounds for developing a hybrid continuum-continuum method for predicting the heat flux and skin friction on blunt bodies in a hypersonic gas flow at any Kn_∞ numbers, which should be based on the use of only continuum flow models as an alternative to continuum-kinetic methods, that require considerably greater computational resources.

7. Conclusion

We have derived asymptotically correct boundary conditions on the surface for the full and thin viscous shock layer models in the two-dimensional problem of the hypersonic flow of a rarefied gas over blunt bodies for low Reynolds numbers.

We have given a simple analytical solution for the skin friction coefficient, the heat-transfer coefficient and the pressure coefficient-on the surface as functions of the free-stream parameters and the geometry of the body in the flow field. This solution is fairly accurate at $Kn_\infty \geq 0.1$ for the heat transfer coefficient, and at $Kn_\infty \geq 0.5$ for the skin friction coefficient in the stagnation region, and the greater the Kn_∞ value the further the distances from the stagnation point where this solution is correct, approaching the solution in free-molecule flow as $Kn_\infty \to \infty$ (for unit accommodation coefficient).

The region of applicability of the full viscous shock layer model for predicting the heat transfer in the windward region of a cold blunt body, moving at a hypersonic velocity is extended when the slip velocity and the temperature jump are taken into account, up to altitudes ~140–150 km of the Space Shuttle reentry trajectory (for a nose radius ~1 m), or up to Kn_∞~15–20. The thin viscous shock layer model gives reliable results for the heat transfer coefficient and the skin friction coefficient in the transitional from regime continuum to free-molecule flow at altitudes above 100 km (for a nose radius ~1 m), or $Kn_\infty \geq 0.1$, and, for these coefficients, ensures a correct limiting transition to values in the free-molecule flow as the flight altitude increases, or as the Knudsen number increases, when $Kn_\infty \to \infty$. At $0.1 \leq Kn_\infty \leq 10$ both models give reliable results, and hence one can use the full viscous shock layer model as the more accurate model for low Kn_∞ numbers, and at Kn_∞~1 one can change to the thin viscous shock layer model as being more correct at high Kn_∞ numbers to predict the heat flux and skin friction.

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References

1. Brykina IG, Rogov BV, Tirskiy GA. Continuum models of rarefied gas flows in problems of hypersonic aerothermodynamics. *Prikl Mat Mekh* 2006;**70**(6):990–1016.
2. Cheng HK. Hypersonic shock-layer theory of the stagnation region at low Reynolds number. In: *Proc. 1961 Heat Transfer and Fluid Mech. Inst.*, Stanford, CA: Stanford Univ. Press.; 1961. pp. 161–175.
3. Cheng H.K. The blunt body problem in hypersonic flow at low Reynolds number, IAS Paper. 1963. No. 63-92. 100 p.
4. Davis RT. Numerical solution of the hypersonic viscous shock layer equation. *AIAA Journal* 1970;**8**(5):843–51.
5. Tirskiy GA. Continuum models in problems of hypersonic rarefied gas flow over blunt bodies. *Prikl Mat Mekh* 1997;**61**(6):903–30.
6. Rogov BV, Sokolova IA. Hyperbolic approximation of the Navier-Stokes equations for mixed viscous flows. *Izv Ross Akad Nauk MZhG* 2002;**3**:20–49.
7. Rogov BV, Tirskly GA. The accelerated method of global iterations for solving the external and internal problem of aerothermodynamics. In: *Proc. 4th Europ. Symp. on Aerothermodynamics for Space Vecicles. 2001*. 2002. p. 537–44.
8. Brykina IG. Asymptotic solution of the two-dimensional equations of a thin viscous shock layer in a rarefied gas for a cold surface. *Izv Ross Akad Nauk MZhG* 2006;**4**:164–72.
9. Brykina IG. Asymptotic solution of the two-dimensional equations of a thin viscous shock layer in a rarefied gas. *Izv Ross Akad Nauk MZhG* 2007;**5**:154–64.
10. Shidlovskii VP. *Introduction to Rarefied Gas Dynamics*. Moscow: Nauka; 1965. pp. 11–14.
11. Hayes WD. *Probstein R. F. Hypersonic Flow Theory*. New York: Acad. Press; 1959.
12. Moss J.N., Cuda V.J. Simmonds A.L. Nonequilibrium effects for hypersonic transitional flow, AIAA Paper, 1987, No. 87-04004. 17p.
13. Moss J.N., Bird G.A. Direct simulation of transitional flow for hypersonic reentry conditions. AIAA Paper. 1984. No. 84-0223. 14 p.
14. Moss J.N., Bird GA. Monte Carlo simulations in support of the Shuttle upper atmosphere mass spectrometer experiment. AIAA Paper. 1985. No. 85-0968.13p.

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